

Deformed Instantons

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In this talk, instantons are discussed in the presence of Lorentz violation. Conventional topological arguments are applied to classify the modified solutions to the Yang-Mills equations according to the topological charge. Explicit perturbations to the instantons are calculated in detail for the case of unit topological charge.

1 Introduction

Yang-Mills theories are typically constructed using a compact Lie Group G with Lie algebra $L(G)$, and a Lie algebra valued vector field $A^\mu(x)$. The action of the group on the vector field is defined by

$$A^\mu(x) \rightarrow U(x)A^\mu(x)U^{-1}(x) - \frac{i}{g}U(x)\partial^\mu U^{-1}(x) \quad , \quad (1)$$

where U is a group element. The field strength tensor is then defined as

$$F^{\mu\nu} = -\frac{i}{g}[D^\mu, D^\nu] \quad , \quad (2)$$

where the covariant derivative is taken as $D^\mu = \partial^\mu + igA^\mu$. With this definition, the field strength transforms in the simple way

$$F^{\mu\nu} \rightarrow U(x)F^{\mu\nu}U^{-1}(x) \quad . \quad (3)$$

The standard gauge invariant action is constructed by forming the integrated trace over the Lorentz invariant square of the field tensor

$$S_{YM}(A) = \frac{1}{2} \int d^4x \operatorname{Tr}[F_{\mu\nu}F^{\mu\nu}] \quad . \quad (4)$$

Extremization of this action with respect to A yields the equations of motion

$$[D_\mu, F^{\mu\nu}] = 0 \quad . \quad (5)$$

The Bianchi identity follows from the definition of the field strength and yields a further set of equations

$$[D_\mu, \tilde{F}^{\mu\nu}] = 0 \quad , \quad (6)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual of F . Note that nonabelian groups yield nonlinear differential equations due to the nonvanishing field commutators.

It is possible that more fundamental theories of nature may contain small Lorentz-breaking effects arising from new physics at higher energy scales.[1] The Standard Model Extension (SME) provides a general framework within which to study Yang-Mills theory in the presence of Lorentz violation.[2, 3] This type of gauge theory has also been extended to include the gravitational sector.[4] Including only gauge invariant and power-counting renormalizable corrections to the Yang-Mills sector yields the action

$$S(A) = \frac{1}{2} \int Tr \left[(F_{\mu\nu}F^{\mu\nu}) + (k_F)^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + (k_{AF})^\kappa \epsilon_{\kappa\lambda\mu\nu} (A^\lambda F^{\mu\nu} - \frac{2}{3}igA^\lambda A^\mu A^\nu) \right] \quad ,$$

where k_F and k_{AF} are constant background fields that parameterize the Lorentz violation. The k_{AF} terms present theoretical difficulties involving negative energy issues[5] even in the abelian case and are therefore neglected in the following analysis. The specific results discussed in this proceedings are derived in more detail elsewhere.[6]

2 Conventional Instantons

In the standard case, static solutions to the Yang-Mills equations with nontrivial, finite action (called instantons[7]) only occur when there are four spatial dimensions.[8] Therefore, it is convenient to transform the standard action to four-dimensional Euclidean space to perform the analysis:

$$S_0(A) = \frac{1}{2} \int d^4x Tr[F^{\mu\nu}F^{\mu\nu}] \quad , \quad (7)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$ is the explicit form of the field tensor. It is convenient to define a quantity called the topological charge q as

$$q = \frac{g^2}{16\pi^2} \int d^4x Tr \tilde{F}^{\mu\nu} F^{\mu\nu} \quad , \quad (8)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual of F . Using the identity $\frac{1}{4}Tr\tilde{F}F = \partial^\mu X^\mu$ with

$$X^\mu = \frac{1}{4}\epsilon^{\mu\nu\lambda\kappa}Tr(A^\nu F^{\lambda\kappa} - \frac{2}{3}igA^\nu A^\lambda A^\kappa) \quad , \quad (9)$$

converts the integral to a surface integral. The net result is that q must be an integer that represents the winding number of the group on the Euclidean three-sphere at infinity. Note that this argument is independent of the explicit form

of the action and only depends on the asymptotic behavior of the fields. The Euclidean version of the equations of motion (5) and the Bianchi Identity (6) yield a set of nonlinear coupled differential equations for A^μ . A clever argument for solving these equations [9] has been developed. The key identity in obtaining the instanton solutions is

$$\frac{1}{2} \int d^4x \text{Tr}(F \mp \tilde{F})^2 \geq 0 \quad . \quad (10)$$

This can be rearranged to yield the condition

$$S \geq \pm \frac{1}{2} \int d^4x \text{Tr}[\tilde{F}^{\mu\nu} F^{\mu\nu}] = \pm \frac{8\pi^2}{g^2} q \quad . \quad (11)$$

The inequality is saturated when the field strength satisfies the duality condition $F = \pm \tilde{F}$. This means that if a self-dual (or anti-self-dual) field strength can be found, it will automatically extremize the action and provide a solution to the equations of motion.

A theorem by Bott[10] states that any mapping of the Euclidean three-sphere into the group can be continuously deformed into a mapping onto an SU(2) subgroup. It is therefore sufficient to consider SU(2) subgroups of the full Lie group G . An explicit example of a self-dual solution for $q = 1$ is given by

$$A_{SD}^\mu = -\frac{\tau^{\mu\nu} x^\nu}{g(\rho^2 + x^2)} \quad , \quad (12)$$

with associated field strength

$$F_{SD}^{\mu\nu} = \frac{2\rho^2}{g(\rho^2 + x^2)^2} \tau^{\mu\nu} \quad , \quad (13)$$

where $\tau^{0i} = \sigma^i$ and $\tau^{ij} = \epsilon^{ijk} \sigma^k$ are expressed in terms of the conventional Pauli matrices. The anti-self-dual solution is the parity transform of the above solution. Subsequent to this, all minimal action solutions have been classified[11] and formally constructed.

3 Deformed Instantons

When Lorentz violation is present, the Euclidean action is modified as

$$S(A) = \frac{1}{2} \int \text{Tr}[(F^{\mu\nu} F^{\mu\nu}) + (k_F)^{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}] \quad . \quad (14)$$

Standard arguments demonstrate that instantons only exist in four Euclidean dimensions, as in the conventional case. The Lorentz violation is assumed small, therefore only leading order contributions from k_F are retained in the following analysis.

As mentioned in the previous section, the topological charge q is an integer, regardless of the form of the action. A modified bound on the action can be derived as

$$S \geq \pm \frac{8\pi^2}{g^2} q \pm \frac{1}{4} (k_F^{\mu\nu\alpha\beta} + \tilde{k}_F^{\mu\nu\alpha\beta}) \int \text{Tr} \tilde{F}^{\mu\nu} F^{\alpha\beta} \quad , \quad (15)$$

where $\tilde{k}_F^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{\mu\nu\lambda\kappa} k_F^{\lambda\kappa\rho\sigma} \epsilon^{\rho\sigma\alpha\beta}$ is a generalized dual to k_F . This expression indicates the natural decomposition $k_F = k_F^+ \oplus k_F^-$ into its self-dual and anti-self-dual parts.

3.1 Case 1: $k_F = -\tilde{k}_F$

This condition implies that k_F takes the form $k_F^{\mu\nu\alpha\beta} = \Lambda_k^{[\mu\alpha} \delta^{\nu]\beta]}$ where $\Lambda_k^{\mu\nu} = \frac{1}{2} k_F^{\alpha\mu\alpha\nu}$ depends only on the trace components of k_F . The action can be minimized using the modified duality condition

$$F' \simeq \pm \tilde{F}' \quad , \quad (16)$$

where $F'^{\mu\nu} = F^{\mu\nu} + \frac{1}{2} k_F^{\mu\nu\alpha\beta} F^{\alpha\beta}$. The explicit solutions can be constructed using the skewed coordinates $\tilde{x}^\mu = x^\mu + \Lambda_k^{\mu\nu} x^\nu$ and writing $A^\mu(x) \simeq A_{SD}^\mu(\tilde{x}) + \Lambda^{\mu\nu} A_{SD}^\nu(x)$ in terms of the modified coordinates. These solutions therefore correspond to conventional instantons in a skewed coordinate system. Note that this is a result of the existence of field redefinitions that can be used to transform physical effects of this type between the fermion and Yang-Mills sectors.[12]

3.2 Case 2: $k_F = \tilde{k}_F$

In this case, k_F is trace free and there is no obvious modified duality condition on F because the lower bound on the action given in Eq.(15) varies with small fluctuations δF , therefore the equations of motion must be solved directly. To find the deformation of the conventional instanton solution, the potential is expanded as $A = A_{SD} + A_k$, where A_{SD} is the conventional self-dual solution and A_k is the unknown perturbation. The linearized equation of motion for A_k is

$$[D_{SD}^\nu, [D_{SD}^\nu, A_k^\mu]] + 2ig[F_{SD}^{\mu\nu}, A_k^\nu] = j_k^\mu \quad , \quad (17)$$

where $j_k^\mu \equiv k_F^{\mu\nu\alpha\beta} [D_{SD}^\nu, F_{SD}^{\alpha\beta}]$ is a set of known functions. This gives a set of linear second-order elliptic differential equations that can be formally solved using propagator techniques:

$$A_k = \int d^4y G(x, y) j_k(y) \quad , \quad (18)$$

where G is the appropriate Green's function.

As an explicit example, consider the deformation of a $q = 1$ instanton in $\text{SU}(2)$. In this case, the direct Green's function approach is unwieldy, therefore the following procedure was eventually adopted:[6]

- Perform a gauge transformation to the singular gauge using $U(x) = -ix \cdot \tau^\dagger/x$ so that the fields become quadratic in the instanton size.
- Work to lowest order in the instanton size ρ using the approximate Green's function $G^{-1} \simeq 4\pi^2(x-y)^2$, and integrate to find the potential.
- Use the tensorial structure of the resulting solution as an ansatz for general values of ρ :

$$A_k^\mu = \frac{2\rho^2 x^2}{3g} f(x^2) k_F^{\mu\nu\alpha\beta} \tau^{\alpha(\nu} x^{\beta)} \quad (19)$$

Remarkably, this gives a differential equation for the unknown function $f(x^2)$.

- Solve the differential equation for f to determine the perturbation to all orders in ρ .

This solution explicitly preserves the topological charge since the asymptotic fields at infinity and at the origin are unmodified. The structure of the instantons is only perturbed in the intermediate region.

4 Summary

Instantons in the presence of Lorentz violation retain their topological properties, but the detailed solutions are deformed. Explicit solutions have been presented here for the case of unit topological charge. The deformations fall into two cases, one for which a simple redefinition of coordinates provides solutions, and another that requires an explicit solution to a set of linear second-order elliptic differential equations. The explicit solution demonstrates that the instanton is unaltered at the boundaries, but is deformed in the intermediate regions.

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